MCKAY
A3040
A117515

MATHEMATICS OF COMPUTATION, VOLUME 30, NUMBER 135
JULY 1976, PAGES 624–631

The Largest Degrees of Irreducible Characters of the Symmetric Group

By John McKay

Abstract. The largest irreducible degrees and the partitions associated with them are tabulated for the symmetric group Σ_n for n up to 75. Analytic upper and lower bounds are derived for the largest degree.

Introduction. A question has been raised by Bivins and others [2]—namely:

For what irreducible representations of the symmetric group Σ_n does the degree attain its maximal value and how does this maximum behave for large n?

This was apparently motivated by the practical considerations of number overflow in the computer but the same question arises in connection with sorting [1].

Each irreducible representation is associated with a partition $a = (a_1, a_2, \ldots, a_k)$, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > 0$, of n. (We shall use $a \in n$ to mean that a is one of the p_n partitions of n.) Its degree is given by [6, p. 61]:

$$d_a = n! \frac{\Pi_{i<j}(b_i - b_j)}{\Pi_i (b_i!)}$$

where $b_i = a_i + k - i$. A combinatorial interpretation of d_a is that it is the number of ways the votes for k candidates can be counted one at a time such that the final total number of votes cast is n and at all stages in the counting $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$, where n_i is the current number of votes for candidate i $(i = 1, \ldots, k)$ with finally $n_i = a_i$ $(i = 1, \ldots, k)$.

Computation of the Maximal Degree. The calculations were made at Edinburgh University on a 4K 12-bit word length PDP8 computer using a multi-length routine for expansile integer multiplication. The strategy is straightforward. For increasing n, partitions of n are generated in natural order (n first and 1^n last) as described in [11]. If a partition, a, precedes or coincides with its conjugate then the degree d_a is computed as in the procedure *degree* of [9] but exponent arithmetic is used retaining integers throughout and avoiding unnecessary overflow. A description of exponent arithmetic appears in [10] but this description is slightly different from that used in this application, and the algorithm given there is a little garbled.

Three arrays are declared, *ex, hfac, lfac* [2: N], where N is the largest integer occurring as a natural factor (here N is at most 75); $ex[n]$ contains the exponent of n in the result and for all $n \leqslant N$, $hfac[n]$ contains the largest prime factor of n and $lfac[n]$ contains the other factor. After initialization, the expression is evaluated by modifying the exponents in *ex*. For example, to divide by $k!$:

for $i := 2$ **step** 1 **until** k **do** $ex[i] := ex[i] - 1$;

Received June 5, 1975; revised September 30, 1975.
AMS (MOS) subject classifications (1970). Primary 20–04, 20C30; Secondary 05A15.
Key words and phrases. Irreducible representation, symmetric group, largest degree.

A3040

Maximal Irreducible Degrees of the Symmetric Group

m_n	n	m_n	n
1	1	247 48435 71200	26
1	2	1276 09121 64000	27
2	3	5742 41047 38000	28
3	4	29528 41929 42320	29
6	5	1 86513 49218 90240	30
16	6	9 24182 73851 90400	31
35	7	50 38573 19942 59200	32
90	8	268 40130 62455 29600	33
216	9	1579 81237 60723 20000	34
768	10	7821 85911 50700 00000	35
2310	11	40971 64298 37000 00000	36
7700	12	2 22250 51347 85087 15200	37
21450	13	15 92694 28320 99526 65600	38
69498	14	93 35226 29027 57090 91840	39
2 92864	15	589 65081 68506 18031 30880	40
11 53152	16	3660 86379 16673 31465 21600	41
48 73050	17	24558 61544 13590 64616 32000	42
163 36320	18	1 40647 43140 34029 84224 96480	43
646 64600	19	8 26287 24406 18222 00507 44960	44
2494 20600	20	50 02839 28761 42234 84343 20000	45
11189 39184	21	309 91868 81321 01700 50024 84000	46
54628 65408	22	2036 88735 12400 42742 34055 68000	47
2 85421 58568	23	13910 87091 49402 51649 95795 35360	48
11 74870 79424	24	1 00788 28728 27294 45059 89182 25920	49
54 75915 90000	25	7 21304 41781 17167 52220 04203 52000	50

m_n	n
54 86245 62826 89907 32913 48475 90400	51
360 27173 44007 80906 66116 28632 57600	52
2416 32801 79788 35907 70622 12235 61800	53
16032 08919 82658 76501 24498 76481 40000	54
1 12332 94008 00148 07351 23185 00477 31500	55
7 80924 18237 44344 89607 49414 47168 50000	56
57 59492 68858 65309 68032 60594 83410 40000	57
392 04228 54325 17105 67342 81079 91024 00000	58
2843 60991 01639 97708 94957 04013 43897 60000	59
23219 99844 17184 55788 71179 66465 14524 16000	60
1 98964 36084 33813 49744 27586 95268 29039 61600	61
14 84932 70650 29909 32159 91941 84305 99280 64000	62
112 80848 15471 49092 37752 38783 18899 58910 11200	63
822 90818 64439 40221 23814 78702 63130 68681 13280	64
6474 45118 59060 42071 22906 42354 58681 10615 19360	65
49264 88872 06925 77842 72444 27860 67020 29690 57200	66
4 02557 12513 54748 85330 10840 14788 82368 98346 54000	67
30 47316 79121 25109 10697 47261 28840 64586 73715 20000	68
234 41791 16438 06987 94867 83935 00955 83550 21660 16000	69
1788 61125 56865 99443 44127 54238 97069 70842 13760 00000	70
14061 79814 66342 15100 92845 75298 46541 20312 24000 00000	71
1 30752 27432 79523 21538 98976 09524 06388 52853 50440 96000	72
10 99941 83391 42975 66548 10097 63063 04543 75434 51852 80000	73
93 93814 29772 20073 46466 22546 26656 28282 24403 04999 04000	74
755 91730 44948 11890 68765 20714 81759 17862 44539 84930 00000	75

A117515

n	$\sqrt{n!}$ = $a.10^b$ a	b	$m_n/\sqrt{n!}$	partition
1	1.00	0	1.000	1
2	1.41	0	0.707	2
3	2.45	0	0.816	2,1
4	4.90	0	0.612	3,1
5	1.10	1	0.548	$3,1^2$
6	2.68	1	0.596	3,2,1
7	7.10	1	0.493	4,2,1
8	2.01	2	0.448	$4,2,1^2$
9	6.02	2	0.359	$4,3,1^2$
10	1.90	3	0.403	4,3,2,1
11	6.32	3	0.366	5,3,2,1
12	2.19	4	0.352	$5,3,2,1^2$
13	7.89	4	0.272	$5,4,2,1^2$
14	2.95	5	0.235	$6,4,2,1^2$
15	1.14	6	0.256	5,4,3,2,1
16	4.57	6	0.252	6,4,3,2,1
17	1.89	7	0.258	$6,4,3,2,1^2$
18	8.00	7	0.204	$7,4,3,2,1^2$
19	3.49	8	0.185	$7,5,3,2,1^2$
20	1.56	9	0.160	$7,5,3,2^2,1$
21	7.15	9	0.157	$7,5,3,2^2,1^2$
22	3.35	10	0.163	7,5,4,3,2,1
23	1.61	11	0.178	$7,5,4,3,2,1^2$
24	7.88	11	0.149	$8,5,4,3,2,1^2$
25	3.94	12	0.139	$8,6,4,3,2,1^2$

n	$\sqrt{n!} = a.10^b$ a	b	$m_n/\sqrt{n!}$	partition
26	2.01	13	0.123	$8,6,4,3,2,1^3$
27	1.04	14	0.122	$8,6,4,3,2^2,1^2$
28	5.52	14	0.104	$8,6,5,3,2^2,1^2$
29	2.97	15	0.099	$8,6,5,4,3,2,1$
30	1.63	16	0.115	$8,6,5,4,3,2,1^2$
31	9.07	16	0.102	$9,6,5,4,3,2,1^2$
32	5.13	17	0.098	$9,7,5,4,3,2,1^2$
33	2.95	18	0.091	$9,7,5,4,3,2,1^3$
34	1.72	19	0.092	$9,7,5,4,3,2^2,1^2$
35	1.02	20	0.077	$9,7,6,4,3,2^2,1^2$
36	6.10	20	0.067	$9,7,6,4,3^2,2,1^2$
37	3.71	21	0.060	$10,8,6,4,3,2^2,1^2$
38	2.29	22	0.070	$9,7,6,5,4,3,2,1^2$
39	1.43	23	0.065	$10,7,6,5,4,3,2,1^2$
40	9.03	23	0.065	$10,8,6,5,4,3,2,1^2$
41	5.78	24	0.063	$10,8,6,5,4,3,2,1^3$
42	3.75	25	0.066	$10,8,6,5,4,3,2^2,1^2$
43	2.46	26	0.057	$11,8,6,5,4,3,2^2,1^2$
44	1.63	27	0.051	$11,8,6,5,4,3,2^2,1^3$
45	1.09	28	0.046	$11,9,7,5,4,3,2^2,1^2$
46	7.42	28	0.042	$11,9,7,5,4,3,2^2,1^3$
47	5.09	29	0.040	$10,8,7,6,5,4,3,2,1^2$
48	3.52	30	0.039	$11,8,7,6,5,4,3,2,1^2$
49	2.47	31	0.041	$11,9,7,6,5,4,3,2,1^2$
50	1.74	32	0.041	$11,9,7,6,5,4,3,2,1^3$

n	$\sqrt{n!}$ = a	$.10^b$ b	$m_n/\sqrt{n!}$	partition
51	1.25	33	0.044	$11,9,7,6,5,4,3,2^2,1^2$
52	8.98	33	0.040	$12,9,7,6,5,4,3,2^2,1^2$
53	6.54	34	0.037	$12,9,7,6,5,4,3,2^2,1^3$
54	4.80	35	0.033	$12,10,8,6,5,4,3,2^2,1^2$
55	3.56	36	0.032	$12,10,8,6,5,4,3,2^2,1^3$
56	2.67	37	0.029	$12,10,8,6,5,4,3^2,2,1^3$
57	2.01	38	0.029	$12,10,8,6,5,4,3^2,2^2,1^2$
58	1.53	39	0.026	$12,10,8,7,5,4,3^2,2^2,1^2$
59	1.18	40	0.024	$12,10,8,7,6,5,4,3,2,1^2$
60	9.12	40	0.025	$12,10,8,7,6,5,4,3,2,1^3$
61	7.12	41	0.028	$12,10,8,7,6,5,4,3,2^2,1^2$
62	5.61	42	0.026	$13,10,8,7,6,5,4,3,2^2,1^2$
63	4.45	43	0.025	$13,10,8,7,6,5,4,3,2^2,1^3$
64	3.56	44	0.026	$13,10,9,7,6,5,4,3,2^2,1^3$
65	2.87	45	0.023	$13,11,9,7,6,5,4,3,2^2,1^3$
66	2.33	46	0.021	$13,11,9,7,6,5,4,3^2,2,1^3$
67	1.91	47	0.021	$13,11,9,7,6,5,4,3^2,2^2,1^2$
68	1.57	48	0.019	$14,11,9,7,6,5,4,3^2,2^2,1^2$
69	1.31	49	0.018	$14,11,9,7,6,5,4,3^2,2^2,1^3$
70	1.09	50	0.016	$14,11,9,8,6,5,4,3^2,2^2,1^3$
71	9.22	50	0.015	$14,11,9,8,6,5,4^2,3,2^2,1^3$
72	7.83	51	0.017	$13,11,9,8,7,6,5,4,3,2^2,1^2$
73	6.69	52	0.016	$14,11,9,8,7,6,5,4,3,2^2,1^2$
74	5.75	53	0.016	$14,11,9,8,7,6,5,4,3,2^2,1^3$
75	4.98	54	0.015	$14,11,10,8,7,6,5,4,3,2^2,1^3$

When complete, the result is reduced to a product of prime powers by decomposing the factors in decreasing order of magnitude into prime factors. The final numerical result may then be obtained. The computation of the cumulated product is speeded up by storing those prime powers that can be contained in a single-length word (viz. up to $2^{12} - 1$). An ALGOL algorithm for the reduction is given by:

```
comment den is not needed if the result is known to be an integer, otherwise the result
is given by num/den;
num := den := 1;
for k := N step −1 until 2 do
      begin if ex[k] ≠ 0 then
            begin if lfac[k] > 1 then
                  begin
                  ex[hfac[k]] := ex[hfac[k]] + ex[k];
                  ex[lfac[k]] := ex[lfac[k]] + ex[k];
                  ex[k] := 0; go to a
                  end;
            if ex[k] > 0 then num := num × k ↑ ex[k];
            if ex[k] < 0 then den := den × k ↑ ex[k]
            end else
a: end
```

The partial product *num* is stored as a multi-length integer and repeatedly multiplied by single-length integers to obtain the final value for the degree d_a. The machine used had no hardware multiplication and the worst case $(2^{12} - 1 \times 2^{12} - 1)$ single-length $\times$ single-length multiplication took approximately ½ millisec = 500 instructions to complete.

The degrees were printed in decimal using Lunnon's [8] multi-length arithmetic package for Atlas.

The tables extend those of Comét [4] (up to $n = 30$) and those of Baer and Brock [1] (up to $n = 36$). They do not seem to reveal any simple recurrence between the partitions associated with the maximal degree for Σ_n and those for Σ_k $(k < n)$. It is notable, however, that frequently a partition for the maximal degree for Σ_n differs from that for the maximal degree for Σ_{n-1} in a single part only.

Bounds for max d_a. Upper and lower bounds for $m_n = \max_{a \in n} d_a$ are easy to find using group-theoretic facts concerning the characters of Σ_n. For a lower bound we have [5, p. 23] that the mean value of d_a is given by s_n/p_n where s_n is the number of solutions to $x^2 = 1, x \in \Sigma_n$; viz.,

$$s_n = \sum_{k=0}^{[\frac{1}{2}n]} \frac{n!}{2^k k!(n-2k)!}.$$

The character column orthogonality relation on the degrees gives $n! = \Sigma_{a \in n} d_a^2$; hence, $s_n/p_n \leqslant m_n \leqslant (n!)^{\frac{1}{2}}$.

Now Chowla, Herstein, and Moore [3] give the asymptotic $s_n \sim 2^{-1/2}(n/e)^{1/2 n}e^{n^{1/2}-1/4}$ and this, together with well-known approximations for p_n and $n!$, gives

$$l_n = \frac{4n\sqrt{3}}{e^{(\pi/3)\sqrt{6n}}} \cdot \frac{(n/e)^{1/2 n}e^{n^{1/2}}}{2^{1/2}e^{1/4}} \leqslant m_n \leqslant (2\pi n)^{1/4}(n/e)^{1/2 n} = u_n$$

from which

$$\frac{u_n}{l_n} = \frac{ke^{tn^{1/2}}}{n^{3/4}}, \quad k = (\pi e/288)^{1/4}, \quad t = \frac{\pi\sqrt{6}}{3} - 1.$$

Remark. The referee has brought to my attention the work of Logan and Shepp [7] who have solved a continuous analogue of this problem and find their result not inconsistent with the partition tabulated here for $n = 75$.

Added in Proof. It has been conjectured from the tables given here that $m_n \leqslant \sqrt{n!}/n$, but Eric Regener has computed that the smallest value of n for which the conjecture is false is $n = 81$ which has a maximal partition of 15, 12, 10, 9, 7, 6, 5, 4, 3^2, 2^2, 1^3.

Department of Computer Science
Concordia University
1455 Maisonneuve West
Montreal, Quebec, Canada

1. R. M. BAER & P. BROCK, "Natural sorting over permutation spaces," *Math. Comp.*, v. 22, 1968, pp. 385–410. MR 37 #3800.

2. R. L. BIVINS, N. METROPOLIS, P. R. STEIN & M. B. WELLS, "Characters of the symmetric groups of degree 15 and 16," *Math. Comp.*, v. 8, 1954, pp. 212–216. MR 16, 333.

3. S. CHOWLA, I. N. HERSTEIN & W. K. MOORE, "On recursions connected with symmetric groups. I," *Canad. J. Math.*, v. 3, 1951, pp. 328–334. MR 13, 10.

4. S. COMÉT, "Improved methods to calculate the characters of the symmetric group," *Math. Comp.*, v. 14, 1960, pp. 104–117. MR 22 #10212.

5. W. FEIT, *Characters of Finite Groups*, Benjamin, New York, 1967. MR 36 #2715.

6. D. E. KNUTH, *The Art of Computer Programming*. Vol. 3, Addison-Wesley, Reading, Mass., 1973.

7. B. F. LOGAN & L. A. SHEPP, "A variational problem for random Young tableaux." (To appear.)

8. W. F. LUNNON, "Multi-length arithmetic for Atlas." (Unpublished.)

9. J. McKAY, "Symmetric group characters–Algorithm 307," *Comm. ACM*, v. 10, 1967, p. 451; ibid., v. 11, 1968, p. 14.

10. J. McKAY, "On the evaluation of multiplicative combinatorial expressions," *Comm. ACM*, v. 11, 1968, p. 392.

11. J. McKAY, "Partitions in natural order–Algorithm 371," *Comm. ACM*, v. 13, 1970, p. 52.